23

# DETECTION OF INVARIANCE, TOTAL SYMMETRY AND PARTIAL SYMMETRY OF SWITCHING FUNCTIONS

SUNIL RANJAN DAS

INSTITUTE OF RADIO PHYSICS AND ELECTRONICS,
UNIVERSITY OF CALCUTTA

(*Received September* 6, 1962)

**ABSTRACT.** The paper suggests a method of identification of total or partial symmetry of a switching function based on the application of the principle of residue test. The invariance of a switching function under a single permutation of two variables can be readily detected from a knowledge of the equality of some of the residues of expansion about those two variables. This procedure of identifying invariance under a single interchange of two variables is then directly extended and applied for the identification of total or partial symmetry of a switching function, the variables of symmetry of which may be either all unprimed, all primed or of mixed nature The method claims the additional advantage of giving all the relevant informations regarding the various alternative representations of symmetries of a switching function with the corresponding a-numbers

## INTRODUCTION

A switching function of $n$ variables which might be either all unprimed, all primed or mixed is said to possess total symmetry in these variables if any interchange of the variables leaves the function identically the same, i e., invariant. A switching function which remains invariant under any interchange of variables which also might be all unprimed, all primed or mixed, belonging to a subset of the set of $n$ variables is said to exhibit partial symmetry in these variables of the subset Importance of detection and recognition of total or partial symmetry associated with a switching function lies in the method of special economical synthesis procedure for the realisation of such functions (Shannon, 1938 and 1949; Washburn, 1949, Keister, Ritchie and Washburn, 1951; G Epstein, 1958).

Methods have been suggested by several authors for the detection of total symmetries of switching functions (Caldwell, 1954; Marcus, 1956; McCluskey, Jr., 1956; T. Singer, 1957; Choudhury and Basu, 1962; Mukhopadhay, 1962). The method described by Caldwell consists in detecting and recognising total symmetry of switching functions by observing certain definite patterns in an extended Karnaugh map (Veitch, 1952; Karnaugh, 1953). The chief disadvantage of this method is that for functions of variables exceeding four, the recognition of symmetries becomes quite difficult. The methods of detection as suggested by Marcus and McCluskey can be applied to switching functions of any number of variables. But these methods do not give any suggestion as to the various alternative forms of representation of symmetries with the corresponding a-numbers of a given switching function. Singer suggested the use of a set of decomposition charts (or "symmetry charts" as he called them) for the detection of total or partial symmetry of switching functions under all permutations of switching variables.

This detection of invariant patterns on the decomposition charts as suggested by Singer is analogous to the recognition of the properties of the residues of expansion of the given switching function about the row variables or column variables. The invariance of a switching function under a single interchange of two variables is readily detected from the equality of the definite groups of residues of expansion about these two variables. The purpose of the present paper is to suggest and develop a method based on the residue test of Boolean functions by numerical methods (Mullin and Kellner, 1955) for the detection of all types of symmetries of switching functions. To detect total symmetry we require only comparison of residues of expansion associated with $n$ expansions. These expansions will also give us all the alternative forms of representation of symmetries of the function with the corresponding $a$-numbers. By utilising this principle of expansion and comparison of the residues of expansion for equality by numerical methods, the partial symmetry of a switching function with respect to the variables which might be all unprimed, all primed or mixed, can also be detected. Alternative representations of partial symmetries, if any, can be found out by this same method.

## INVARIANCE OF A SWITCHING FUNCTION UNDER A SINGLE INTERCHANGE OF TWO VARIABLES

Any switching function $F(x_{n-1}, x_{n-2}, \ldots, x_0)$ of $n$ binary variables $x_{n-1}, x_{n-2}, \ldots, x_0$, when expanded about any two variables, let us say, $x_{n-1}$ and $x_{n-2}$, can be written as

$$F(x_{n-1}, x_{n-2}, \ldots, x_0) = x_{n-1}\, x_{n-2}\, R_0 + x_{n-1} x'_{n-2}\, R_1 + x'_{n-1}\, x_{n-2}\, R_2 + x'_{n-1}\, x'_{n-2}\, R_3 \quad \ldots (1)$$

where $R_0$, $R_1$, $R_2$ and $R_3$ are the residual functions of expansions of $(n-2)$ variables, not including $x_{n-1}$ and $x_{n-2}$.

From equation (1) it is seen that for the function $F(x_{n-1}, x_{n-2}, \ldots, x_0)$ to remain invariant under different permutation and negation operations of the two variables $x_{n-1}$ and $x_{n-2}$, the following conditions must be satisfied :

(i) When both the variables $x_{n-1}$ and $x_{n-2}$ are unprimed, for invariance under single permutation, $R_1 = R_2$.

(ii) When both the variables $x_{n-1}$ and $x_{n-2}$ are primed, for invariance under single permutation, $R_0 = R_3$.

(iii) When one of the variables is primed and the other unprimed, that is, when $x_{n-1}$ is primed and $x_{n-2}$ is unprimed or $x_{n-1}$ is unprimed and $x_{n-2}$ is primed, then for invariance under permutation, $R_0 = R_1 = R_2 = R_3$, i.e., all the four residues of expansion are identical. When this last condition is satisfied, it is implied that conditions (i) and (ii) are necessarily satisfied, and $x_{n-1}$ and $x_{n-2}$ can be eliminated from the expression of the function.

Therefore in order to detect the invariance of a switching function under a single interchange of two variables (the variables being either both primed, both unprimed or mixed) we need only expand the function about these two variables and compare the different residues of expansion for equality. This expansion and subsequent comparison of the residues of expansion to detect invariance can be done very effectively by utilising the method of residue test as applied to Boolean functions expressed in decimal mode.

When the residue test is applied to a transmission function expressed as a standard sum it is merely necessary to carry out a process of factoring in order to find the residues Taking a function $F = (x'_3x_2x'_1x_0 + x'_3x_2x_1x_0 + x_3x'_2x_1x_0 + x_3x_2x_1x_0)$, to apply the residue tests to the $x_2$ variable, we may perform the expansion by factoring out $x_2$ and $x'_2$ to obtain $F = x_2(x'_3x'_1x_0 + x'_3x_1x_0 + x_3x_1x_0) + x'_2(x_3x_1x_0)$

In order to evaluate the residues of expansion about any variable of a given switching function by numerical methods, we are to know for the given binary digit position in the binary number representation of the function whether the digit is a zero or an one, corresponding to the primed and unprimed literal in the algebraic expression and then to group the terms to form "1" residue and "0" residue.

This can be done very easily if we note that as in a decimal number shifting of decimal point one digit position to the left is equivalent to dividing the number by 10, so also shifting of binary point one digit position to the left is equivalent to the division of the binary number by 2 Taking a binary number 111101 whose decimal equivalent is 61 and which might be thought to represent one of the terms in a standard sum expression of a given switching function, if we now want to know whether the third digit from the left in this particular binary number is a 0 or an 1, without actually writing the same but from a knowledge of its decimal counterpart, what we should do in fact is as follows (Mullin and Kellner, 1955).

Let us take the binary number 111101 and putting a binary point after the third digit from the left, write it as 111.101, decimal equivalent of which is

$$2^2+2^1+2^0+2^{-1}+2^{-3} = 4+2+1+1/2+1/8 = 7\tfrac{5}{8} \qquad \dots \quad (2)$$

Thus we see that by putting the binary point at the position mentioned, we have practically divided the original binary number by $2^3$ or its equivalent decimal value 61 by 8. Discarding all the binary digits to the right of the binary point we now obtain 111 which is equivalent to the decimal number 7, an integer which can be obtained independently from the number $7\frac{5}{8}$ by discarding the fractional part 5/8. This integral part of the decimal quotient $7\frac{5}{8}$ or 7 is an odd number, a fact which goes to imply that the right hand digit of its binary equivalent (111) is unity. If the whole number part of the quotient had been even, the digit at the extreme right of the equivalent binary number would have been zero.

From this analysis, the general procedure for identifying a primed or an unprimed literal in a term of the standard sum expression (which is the same as to know whether the digit which represents it in the equivalent binary form is a 0 or an 1) of a switching function may be stated thus : We should divide the binary number equivalent of the term by that power of 2 which places the binary point to the right of the digit whose identity is to be disclosed, an operation which is the same as to divide the decimal counterpart of the same binary number by that same power of 2 and to note whether the integral part of the decimal quotient obtained thereby is an even or an odd integer, an even integer denoting a primed and an odd integer an unprimed literal.

*Example* :

To illustrate the principle, let us take an example

$$F(x_3, x_2, x_1, x_0) = \Sigma(4, 6, 8, 12) \tag{3}$$

To apply the residue test to the $x_2$ variable we see that in the binary number representation of the function, the binary point should be shifted two digit positions to the left which requires that its decimal counterpart be divided by 4. Thus dividing by 4, the results obtained for different terms of the transmission are :

$$\frac{4}{4} = 1+,\ \text{odd};\ \frac{6}{4} = 1+,\ \text{odd};\ \frac{8}{4} = 2+,\ \text{even; and}\ \frac{12}{4} = 3+,\ \text{odd}.$$

Hence the residues can be written by grouping the decimal numbers, as

$$F(x_3, x_2, x_1, x_0) = x_2(4, 6, 12) + x'_2(8) \quad \ldots \quad (4)$$

Now considering the decimal integers 12 and 8, we note that for the number 12 represented by the binary number 1100, the $x_2$ literal may be factored out by writing $12 \rightarrow x_2(1\text{--}00)$ and for the number 8 represented by 1000, the $x'_2$ literal may be taken out to indicate by writing $8 \rightarrow x'_2(1\text{--}00)$, thereby showing the $x_2$ and $x'_2$ residues to be identical.

This equality of the residues of expansion can also be shown directly if we note that the residues of the binary numbers 1100 and 1000 would also remain equal if we would replace the "0" in the $x_2$ position of 1000 by "1" and expand both of them about the $x_2$ literal. This replacement of "0" by "1" in the $x_2$ position of 1000 is identical with the addition of $2^2$ to the binary number 1000 or 4 to its equivalent decimal number So in the expansion given by

$$F = x_2(4, 6, 12) + x'_2(8)$$

we get, by adding 4 to 8, $x'_2(12)$ in place of $x'_2(8)$, which gives an easy way of identifying whether the $x'_2$ residue is equal to or contained in the $x_2$ residue The presence of the same term 12 in both the $x_2$ and $x'_2$ residues proves that when $x_2$ is factored out of the algebraic or binary equivalent of the number 12 and $x'_2$ from that of the number 8(by addition of 4 to which we obtain the second 12 in this case) what remains in the bracket (1-00) is the same in both the cases.

By the application of the above principle, expansion about any number of variables can be obtained, first expanding the function about one of the variables, then expanding the residues of first expansion about another variable and so on and lastly the residues can be modified by adding appropriate decimal numbers to the respective residue groups to test for equality.

Let us now apply this principle of expansion and comparison of residue groups to test the invariance of a switching function under single permutation of two variables.

*Example* :

The switching function $F(x_3, x_2, x_1, x_0) = \Sigma(1, 2, 5, 6, 8, 11, 12, 15)$ is invariant under the following permutation and permutation and negation operations :

$$\text{(a) } x_3 \sim x_1, \quad \text{(b) } x'_3 \sim x'_1, \quad \text{(c) } x_3 \sim x_0, \quad \text{(d) } x'_3 \sim x'_0$$

and (e) $x_1 \sim x_0$, (f) $x'_1 \sim x'_0$ where the sign '$\sim$' means 'interchanged with' This can be shown from the expansion of the function about these different pairs of variables and comparing the different residue groups as given below.

$$\begin{aligned} F &= x_3(8, 11, 12, 15) + x'_3(1, 2, 5, 6) \\ &= x_3x_1(11, 15) + x_3x'_1(8, 12) + x'_3x_1(2, 6) + x'_3\,x'_1(1, 5). \end{aligned}$$

On modification, the different residue groups become

$$x_3x_1(11, 15),\ x_3\,x'_1(10, 14),\ x'_3x_1(10, 14),\ x'_3x'_1(11, 15)$$

which shows that the residual functions associated with $x_3x'_1$ and $x_3'x_1$ are equal (i.e., $R_1 = R_2$) as well as those associated with $x_3x_1$ and $x'_3x'_1$ (i.e., $R_0 = R_3$). This means that the function possesses invariance for

$$\text{(a) } x_3 \sim x_1 \quad \text{and} \quad \text{(b) } x'_3 \sim x'_1.$$

Similarly, expanding the function about $x_3x_0$ and $x_1x_0$ and modifying the residue groups give

$$x_3x_0(11, 15),\ x_3x'_0(9, 13),\ x'_3x_0(9, 13),\ x'_3x'_0(11, 15)$$

and
$$x_1x_0(11, 15),\ x_1x'_0(3, 7),\ x'_1x_0(3, 7),\ x'_1x'_0(11, 15)$$

which show that the function is invariant for (c) $x_3 \sim x_0$ and (d) $x'_3 \sim x'_0$, and (e) $x_1 \sim x_0$ and (f) $x'_1 \sim x'_0$. We shall next discuss some further properties associated with the equality of the residual functions of expansion The total or partial symmetry of switching functions will be detected on the basis of these results.

If in an $n$ variable switching function $F(x_{n-1}, x_{n-2}, \ldots, x_0)$, expansion about the pair of variables $x_{n-1}, x_{n-2}$ and comparison of the residual functions of expansion show that $R_1 = R_2$, it not only means that the function is invariant for $x_{n-1}$ permuted with $x_{n-2}$ but it also means that all the functions derived from the given function by applying $(n-2)!\, 2^{n-2}$ transformations (negation and permutation operations associated with $n-2$ variables) to the remaining $(n-2)$ variables will also continue to remain invariant. This is because any transformation applied to these $(n-2)$ variables will alter the residual functions identically so that $R_1$ and $R_2$ will still remain identical. The same will be the case for $x'_{n-1}$ permuted with $x'_{n-2}$ when $R_0 = R_3$. Further if in the expansion of the switching function about $x_{n-1}$ and $x_{n-2}$, the residual functions $R_1 = R_2$ this also will not only mean that the function is invariant for $x_{n-1} \sim x_{n-2}$ but also the function derived from the given function by priming both $x_{n-1}$ and $x_{n-2}$ will continue tor emain invariant when $x'_{n-1} \sim x'_{n-2}$, and this invariance is unaffected by the application of any transformation to the set of $(n-2)$ residual variables. Likewise, when the residual functions $R_0 = R_3$ it also not only means that the function is invariant for $x'_{n-1} \sim x'_{n-2}$ but also the function obtained from the given function by priming either $x_{n-1}$ or $x_{n-2}$ will also continue to remain invariant when $x'_{n-1} \sim x_{n-2}$ or $x_{n-1} \sim x'_{n-2}$ as the case may be, invariance being of course unaffected by any transformation applied to the group of $(n-2)$ residual variables (Mukhopadhyay, 1962). These results may be summarised thus :

(i) If $R_1 = R_2$,

$$F(x_{n-1}, x_{n-2}, \ldots, x_0) \equiv x_{n-1} \sim x_{n-2} \qquad \ldots \quad (5)$$

and
$$\urcorner F(x_{n-1}, x_{n-2}, \ldots, x_0) \equiv x_{n-1} \sim x_{n-2} \qquad \ldots \quad (6)$$

Also
$$F(x'_{n-1}, x'_{n-2}, \ldots, x_0) \equiv x'_{n-1} \sim x'_{n-2} \qquad \ldots \quad (7)$$

and
$$\urcorner F(x'_{n-1}, x'_{n-2}, \ldots, x_0) \equiv x'_{n-1} \sim x'_{n-2} \qquad \ldots \quad (8)$$

(ii) If $R_0 = R_3$,

$$F(x'_{n-1}, x_{n-2}, \ldots, x_0) \equiv x'_{n-1} \sim x_{n-2} \qquad \ldots \quad (9)$$

and
$$\urcorner F(x'_{n-1}, x_{n-2}, \ldots, x_0) \equiv x'_{n-1} \sim x_{n-2} \qquad \ldots \quad (10)$$

and also $$F(x_{n-1}, x'_{n-2}, \ldots, x_0) \equiv x_{n-1} \sim x'_{n-2} \quad (11)$$

and $$\urcorner F(x_{n-1}, x'_{n-2}, \ldots, x_0) \equiv x_{n-1} \sim x'_{n-2} \quad (12)$$

where in the above

'$\equiv$' denotes invariance under transformation,

and '$\urcorner$' denotes different functions obtained by applying $(n-2)!2^{n-2}$ transformations amongst the $(n-2)$ residual variables.

## DETECTION OF TOTAL SYMMETRY

To detect total symmetry of a switching function, it is not necessary to test invariance of the function under every interchange of two variables. it is sufficient to test the invariance of the function for $n$ interchanges of two variables (for an $n$ variable switching function) where the sets of variables form a closed chain. For example, if an $n$ variable switching function $F(x_{n-1}, x_{n-2}, x_{n-3}, \ldots, x_1, x_0)$ is invariant under the permutations $x_{n-1} \sim x_{n-2}, x_{n-2} \sim x_{n-3}, \ldots, x_1 \sim x_0, x_0 \sim x_{n-1}$ then the function is also invariant for any intermediate permutation $x_p \sim x_q$ $(p \neq q)$ and hence totally symmetric because any intermediate permutation can be generated from the above sets of cyclic permutations (Chatterjee, 1957). The aggregate of $n$ expansions about the pairs of variables $(x_{n-1}, x_{n-2})$, $(x_{n-2}, x_{n-3})$, $\ldots$, $(x_1, x_0)$, $(x_0, x_{n-1})$ may be called the set of cyclic expansions and will be utilised to detect total symmetry associated with a switching function as illustrated below.

*Example* :

To show whether the five variable switching function given by

$$F(x_4, x_3, x_2, x_1, x_0) = \Sigma(0, 1, 3, 4, 6, 7, 8, 10, 11, 14, 17, 20, 21, 23, 24, 25, 27, 28, 30, 31) \quad \ldots \quad (13)$$

possesses total symmetry.

Taking the function, it is cyclically expanded. We have (1) from expansion about $x_4x_3$,

$$F = x_4(17, 20, 21, 23, 24, 25, 27, 28, 30, 31) + x'_4(0, 1, 3, 4, 6, 7, 8, 10, 11, 14)$$

$$= x_4x_3(24, 25, 27, 28, 30, 31) + x_4x'_3(17, 20, 21, 23) + x'_4x_3(8, 10, 11, 14) + x'_4x'_3(0, 1, 3, 4, 6, 7).$$

On modification, the above residues become

$$x_4x_3(24, 25, 27, 28, 30, 31),\ x_4x'_3(25, 28, 29, 31),$$

$$x'_4x_3(24, 26, 27, 30),\ x'_4x'_3(24, 25, 27, 28, 30, 31)$$

wherefrom it appears that $R_0 = R_3$.

$$\therefore \quad F(x'_4, x_3, x_2, x_1, x_0) \equiv x'_4 \sim x_3 \qquad \ldots \quad (14)$$

and $$\urcorner\, F(x'_4, x_3, x_2, x_1, x_0) \equiv x'_4 \sim x_3 \qquad \ldots \quad (15)$$

Also $$F(x_4, x'_3, x_2, x_1, x_0) \equiv x_4 \sim x'_3 \qquad \ldots \quad (16)$$

and $$\urcorner\, F(x_4, x'_3, x_2, x_1, x_0) \equiv x_4 \sim x'_3 \qquad \ldots \quad (17)$$

(2) from expansion about $x_3x_2$,

$$F = x_3x_2(14, 28, 30, 31) + x_3x'_2(8, 10, 11, 24, 25, 27) + x'_3x_2(4, 6, 7, 20, 21, 23)$$
$$+ x'_3x'_2(0, 1, 3, 17).$$

On modification, these residues become

$$x_3x_2(14, 28, 30, 31),\ x_3x'_2(12, 14, 15, 28, 29, 31),\ x'_3x_2(12, 14, 15, 28, 29, 31),$$
$$x'_3x'_2(12, 13, 15, 29)$$

wherefrom it appears that $R_1 = R_2$.

Hence $$\urcorner\, F(x_3, x_2, x_4, x_1, x_0) \equiv x_3 \sim x_2 \qquad \ldots \quad (18)$$

and $$\urcorner\, F(x'_3, x'_2, x_4, x_1, x_0) \equiv x'_3 \sim x'_2 \qquad \ldots \quad (19)$$

(3) from expansion about $x_2x_1$,

$$F = x_2x_1(6, 7, 14, 23, 30, 31) + x_2x'_1(4, 20, 21, 28)$$
$$+ x'_2x_1(3, 10, 11, 27) + x_2'x'_1(0, 1, 8, 17, 24, 25).$$

Modifying the residues we have

$$x_2x_1(6, 7, 14, 23, 30, 31),\ x_2x'_1(6, 22, 23, 30),$$
$$x'_2x_1(7, 14, 15, 31),\ x'_2x'_1(6, 7, 14, 23, 30, 31)$$

which shows that $R_0 = R_3$.

$$\therefore \quad \urcorner\, F(x'_2, x_1, x_4, x_3, x_0) \equiv x'_2 \sim x_1 \qquad (20)$$

and $$\urcorner\, F(x_2, x'_1, x_4, x_3, x_0) \equiv x_2 \sim x'_1 \qquad (21)$$

(4) from expansion about $x_1x_0$,

$$F = x_1x_0(3, 7, 11, 23, 27, 31) + x_1x'_0(6, 10, 14, 30) + x'_1x_0(1, 17, 21, 25) + x'_1x'_0(0, 4, 8, 20, 24, 28).$$

On modification, these residues become

$$x_1x_0(3, 7, 11, 23, 27, 31),\ x_1x'_0(7, 11, 15, 31),\ x'_1x_0(3, 19, 23, 27),$$
$$x'_1x'_0(3, 7, 11, 23, 27, 31)$$

which shows that $R_0 = R_3$.

$$\therefore \quad \urcorner \quad F(x'_1, x_0, x_4, x_3, x_2) \equiv x'_1 \sim x_0 \qquad \ldots \quad (22)$$

and
$$\urcorner \quad F(x_1, x'_0, x_4, x_3, x_2) \equiv x_1 \sim x'_0 \qquad \ldots \quad (23)$$

(5) and lastly from expansion about $x_4x_0$,

$$F = x_4x_0(17, 21, 23, 25, 27, 31) + x_4x'_0(20, 24, 28, 30) + x'_4x_0(1, 3, 7, 11) + x'_4x'_0(0, 4, 6, 8, 10, 14).$$

On modification, we have

$$x_4x_0(17, 21, 23, 25, 27, 31),\ x_4x'_0(21, 25, 29, 31),$$
$$x'_4x_0(17, 19, 23, 27),\ x_4'x_0'(17, 21, 23, 25, 27, 31)$$

which shows that $R_0 = R_3$.

$$\therefore \qquad \urcorner\, F(x'_4, x_0, x_3, x_2, x_1) = x'_4 \sim x_0 \qquad \ldots \quad (24)$$

and
$$\urcorner\, F(x_4, x'_0, x_3, x_2, x_1) = x_4 \sim x'_0 \qquad \ldots \quad (25)$$

Combining equations (17, (19), (20), (23), and (25) we have

$$F(x_4, x'_3, x'_2, x_1, x'_0) \equiv x_4 \sim x'_3,\ x'_3 \sim x'_2,\ x'_2 \sim x_1,\ x_1 \sim x'_0 \text{ and } x'_0 \sim x_4$$

So the function is totally symmetric and the variables of symmetry are $x_4$, $x'_3$, $x_2'$, $x_1$, $x'_0$. $a$-numbers of the function can be found by writing the function in the Truth Table form and double negating the columns under $x_3$, $x_2$, and $x_0$. This matrix must have sufficient row sum occurrences, otherwise the chain of cyclic permutations could not have been complete (M. P Marcus, 1956).

The above matrix shows that the function can be written as

$$S_{2,3}(x_4, x'_3, x'_2, x_1, x'_0).$$

TABLE I

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $x_4$ | $x'_3$ | $x'_2$ | $x_1$ | $x'_0$ | Number of ones |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 3 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 3 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 3 |
| 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 2 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 3 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 2 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 3 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 3 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 3 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 2 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 3 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 3 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 2 |
| Ratio of the number of ones to number of zeros. | | | | | $\frac{10}{10}$ | $\frac{10}{10}$ | $\frac{10}{10}$ | $\frac{10}{10}$ | $\frac{10}{10}$ | |

Similarly, combining the sets of equations (15), (18), (21), (22) and (24) the function can be identified as

$$S_{0,1,4,5}(x'_4, x_3, x_2, x'_1, x_0).$$

To find the variables of symmetry associated with a symmetric switching function from the set of cyclic permutations and hence to find all the alternative representations of symmetries of the given switching function, the general procedure may be outlined by taking the example of equation (13) and using the following sets of relations.

$$R_0 = R_3 \longleftrightarrow x'_4 \sim x_3, \;\; x_4 \sim x'$$

$$R_1 = R_2 \longleftrightarrow x_3 \sim x_2, \;\; x'_3 \sim x'$$

$$R_0 = R_3 \longleftrightarrow x'_2 \sim x_1, \;\; x_2 \sim x'$$

$$R_0 = R_3 \longleftrightarrow x'_1 \sim x_0, \;\; x_1 \sim x'$$

$$R_0 = R_3 \longleftrightarrow x'_0 \sim x_4, \;\; x_0 \sim x'.$$

To start with we shall take any literal, preferably one with the highest value of the subscript $p$, both primed and unprimed and then associate them with literals with which they are connected by '$\sim$' signs In the above case, $x_4$ and $x'_4$ are connected with $x'_3$ and $x_3$ respectively. Similarly, $x'_3$ and $x_3$ are connected with $x'_2$ and $x_2$

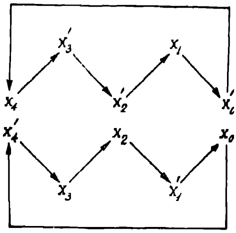


Fig 1. Possible closed paths indicating possible variables of symmetry.

respectively and so on we proceed until we end on the literals with which we started. The literals that occur in any closed path will give a set of variables of symmetry Thus, for the above example, there can be traced two and only two closed paths associated with the two possible representations of the functions as given earlier (Fig. 1).

## DETECTION OF PARTIAL SYMMETRY

To detect and identify partial symmetry of a switching function, we should look for the invariance of the function under any interchange of the variables belonging to a subset of the set of variables. The chain of cyclic permutations will be complete with these variables of the subset, the variables being either all unprimed, or all primed or of mixed type Hence, in this case depending on the nature of the problem, complete set of expansions about two variables might be required. Let us take an example to illustrate the method

*Example* .

Find the partial symmetry associated with the function

$$F(x_3, x_2, x_1, x_0) = \Sigma(0, 1, 3, 4, 6, 7, 9, 10, 12, 15) \tag{26}$$

Taking the function, sets of expansions are done. We have

(i) from expansion about $x_3x_2$,

$$F = x_3(9, 10, 12, 15)+x'_3(0, 1, 3, 4, 6, 7)$$

$$= x_3x_2(12, 15)+x_3x'_2(9, 10)+x'_3x_2(4, 6, 7)+x'_3x'_2(0, 1, 3).$$

The residues, on modification, become

$$x_3x_2(12, 15),\ x_3x'_2\,(13, 14),\ x'_3x_2(12, 14, 15) \text{ and } x'_3x'_2(12, 13, 15)$$

which shows that the residues are not equal.

Similarly, (ii) from expansion about $x_3x_1$ and on modifying the residue groups, we get

$$x_3x_1(10, 15),\ x_3x'_1(11, 14),\ x'_3x_1(11, 14, 15),\ x'_3x'_1(10, 11, 14)$$

which also shows that the residues are not equal.

Likewise (iii) from expansion about $x_3x_0$ and on modifying the residue groups, we get

$$x_3x_0(9, 15),\ x_3x'_0(11, 13),\ x'_3x_0(9, 11, 15),\ x'_3x'_0(9, 13, 15)$$

showing the residues to be unequal.

But by (iv) expanding the function about $x_2x_1$ and modifying the residue groups, we have

$$x_2x_1(6, 7, 15),\ x_2x'_1(6, 14),\ x'_2x_1(7, 14),\ x'_2x'_1(6, 7, 15)$$

which shows that $R_0 = R_3$

$$\therefore \qquad F(x'_2, x_1, x_3, x_0) \equiv x'_2 \sim x_1 \qquad \dots \quad (27)$$

and

$$F(x_2, x'_1, x_3, x_0) \equiv x_2 \sim x'_1 \qquad \dots \quad (28)$$

(v) expanding the function about $x_1x_0$ and modifying the residue groups, we get

$$x_1x_0(3, 7, 15),\ x_1x'_0(7, 11),\ x'_1x_0(3, 11),\ x'_1x'_0(3, 7, 15)$$

which shows that $R_0 = R_3$.

$$\therefore \qquad F(x'_1, x_0, x_3, x_2) \equiv x'_1 \sim x_0 \qquad \dots \quad (29)$$

and

$$F(x_1, x'_0, x_3, x_2) \equiv x_1 \sim x'_0 \qquad \dots \quad (30)$$

and (vi) lastly expanding about $x_2x_0$ and modifying, we have the residues

$$x_2x_0(7, 15),\ x_2x'_0(5, 7, 13),\ x'_2x_0(5, 7, 13),\ x'_2x'_0(5, 15)$$

which shows that $R_1 = R_2$.

$$\therefore \qquad F(x_2, x_0, x_3, x_1) \equiv x_2 \sim x_0 \qquad \dots \quad (31)$$

and

$$F(x'_2, x'_0, x_3, x_1) \equiv x'_2 \sim x'_0 \qquad \dots \quad (32)$$

From the above sets of expansions we see that a closed chain of cyclic permutations comprises the variables $x_2$, $x_1'$, $x_0$ and $x'_2$, $x_1$, $x'_0$ so that the function is partially symmetric with respect to these variables. To write the form of the function, the function should be entered on a matrix and appropriate columns should be double negated. By such operations, the function may be written as

$$x_3S_0(x_2, x'_1, x_0)+x'_3S_1(x_2, x'_1, x_0)+(x_3+x'_3)S_2(x_2, x'_1, x_0) \quad \ldots \quad (33)$$

or

$$x'_3S_2(x'_2, x_1, x'_0)+x_3S_3(x'_2, x_1, x'_0)+(x_3+x'_3)S_1(x'_2, x_1, x'_0) \quad \ldots \quad (34)$$

The function is also partially symmetric with respect to the subsets of variables of the sets $(x_2, x'_1, x_0)$ and $(x'_2, x_1, x'_0)$

## DISCUSSIONS

In this paper a method has been suggested for the detection of invariance and recognition of the symmetries (both total and partial) of switching functions having variables of symmetry which may be either all primed, all unprimed or of mixed nature. The method of detection of total or partial symmetries of switching functions suggested in this paper by the extension of the principle of the residue test readily gives all the alternative representations of symmetries of functions (total or partial) with their corresponding $a$-numbers. Importance of detection of total or partial symmetries of switching functions arises principally of the fact that they lead to relay contact networks or electronic circuits which are more economical of elements (diodes, relay contacts, etc.). Hence the knowledge of all those alternative forms of representations of symmetries is considered very important from the actual circuit synthesis point of view The circuit synthesized from one of these forms may be more desirable than the others from design considerations Hence when all the forms of representations of symmetries are known, an easy comparison amongst them for economy and simplicity of design can be made. Further, since the different representations of symmetries are associated with different priming operations of the variables of the functions, all the different circuits may also be easily obtained if required, from any one, by appropriately complementing the input variables, when one of them is known.

Lastly, the method is not only exhaustive but also simple and straightforward at the same time.

## ACKNOWLEDGMENT

The author wishes to express his indebtedness to Prof. J. N. Bhar, D.Sc., F.N.I., M. Brit. I.R.E , for his guidance and keen interest in the work and to Dr. A. K Choudhury, M.Sc., D.Phil. for discussions and helpful suggestions throughout the progress of the work. The award of a Senior Research Fellowship by the C. I. S. R. Government of India, is also thankfully acknowledged.

## REFERENCES

Ashenhurst, R. L., 1957, *Proc. International Symposium on the Theory of Switching in the Annals of the Computation Laboratory of Harvard University*, Vol. **29**, 74-116.

Caldwell, S. H., 1954, *Trans. A.I.E.E.*, Vol. **73**, Part I, 142-146.

Caldwell, S. H., 1958, *Switching Circuits and Logical Design*, John Wiley and Sons, Inc., N.Y., 236-270.

Chatterjee, B. C, 1957, *Abstract Algebra, Part I*, Das Gupta & Co. (Private) Ltd, Calcutta.

Choudhury, A. K., and Basu, M. S., 1962, *Indian Journal of Physics*, Vol. **36**, No. 1, 31-42

Epstein, G, 1958, *I.R.E Transactions on Electronic Computers*, Vol *EC*-7, No. 1, 57-60.

Karnaugh, M., 1953, *Trans A.I.E.E.*, Vol. **72**, Part I, 593-598.

Keister, W., Ritchie, A E, and Washburn, S H, 1951, *The Design of Switching Circuits*, D. Van Nostrand Co. Inc., New York, N. Y

Marcus, M. P, 1956, *I.R.E. Transactions on Electronic Computers*, Vol. *EC*-5, 237-239.

McCluskey, E. J., Jr, 1956, *Bell System Technical Journal*, Vol **35**, 1445-1453.

Mukhopadhyay, A, 1962, *Studies on Switching Circuit Theory* (Doctoral Thesis).

Mullin, A. A, and Kellner, W G, 1955, "*A Residue Test for Boolean Functions*", Unpublished teaching Memorandum, Department of Electrical Engineering, M.I T.

Shannon, C E, 1938, *Trans. A.I E E.*, Vol **57**, 713-723.

Shannon, C E, 1949, *Bell System Technical Journal*, Vol. **28**, 59-98.

Singer, T, 1957, *Proc. International Symposium on the Theory of Switching in the Annals of the Computation Laboratory of Harvard University*, Vol **29**, 125-133

Veitch, E. W, 1952, *Proc. of Association for Computing Machinery*, 127-133.

Washburn, S H, 1949, *Trans. A.I.E.E*, Vol. **68**, 582-586.